THE COWEN-DOUGLAS CLASS AND DE BRANGES-ROVNYAK SPACES

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Abstract. We establish a connection between the de Branges-Rovnyak spaces and the Cowen-Douglas class of operators which is associated with complex geometric structures. We prove that the backward shift operator on a de Branges-Rovnyak space never belongs to the Cowen-Douglas class when the symbol is an extreme point of the closed unit ball of H^∞ (the algebra of bounded analytic functions on the open unit disk). On the contrary, in the non-extreme case, it always belongs to the Cowen-Douglas class of rank one. Additionally, we compute the curvature in this case and derive certain exotic results on unitary equivalence and angular derivatives.

Contents

1. Introduction

In this paper, we bring together two distinct and classical theories that have so far remained largely unconnected: the Cowen-Douglas class of operators and the de Branges-Rovnyak spaces. Recall that the Cowen-Douglas class of operators is associated with complex geometric structures, specifically hermitian holomorphic vector bundles. Its curvature corresponds to the

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Chern connection for such bundles, and the problem of complete unitary invariance is examined through the equivalence or equality of curvatures. One important aspect of the Cowen-Douglas class of operators is the assumption of an abundance of eigenvectors in terms of rich point spectrums.

On the other hand, the de Branges–Rovnyak spaces are primarily function-theoretic objects whose structure fundamentally depends on the nature of the symbol that parametrizes them. To be more specific, we recall the standard notation in which H^∞ denotes the commutative Banach algebra of all bounded analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The norm on H^∞ is the standard supremum norm $\|\cdot\|_\infty$ on $\mathbb{D}$. Set

$$H_1^\infty = \{b \in H^\infty : \|b\|_\infty \leqslant 1\}.$$

Given a function $b \in H_1^\infty$, the Toeplitz operator T_b is a contraction on the Hardy space H^2. Here, H^2 denotes the Hilbert space of analytic functions on $\mathbb{D}$ whose power series coefficients form a square-summable sequence (also see (2.1)). The *de Branges-Rovnyak space* corresponding to $b \in H_1^\infty$ is the Hilbert space

$$\mathcal{H}(b) = (I - T_b T_{\bar{b}})^{\frac{1}{2}} H^2,$$

equipped with the inner product

$$\langle (I - T_b T_{\bar{b}})^{\frac{1}{2}} f, (I - T_b T_{\bar{b}})^{\frac{1}{2}} g \rangle_b = \langle f, g \rangle_2,$$

where $f, g \in H^2 \ominus \ker(I - T_b T_{\bar{b}})^{\frac{1}{2}}$ and $\langle \cdot, \cdot \rangle_2$ denotes the scalar product in H^2. Note that $T_b^* = T_{\bar{b}}$, and then, since T_b is a contraction, the operator $I - T_b T_{\bar{b}}$ is positive and so the definition above makes sense.

The key point here is that $\mathcal{H}(b)$ is contractively contained in H^2. Following convention, we denote the Toeplitz operator T_z which has the analytic symbol z, as the unilateral forward shift operator S:

$$Sf = T_z(f) = zf, \qquad f \in H^2.$$

Its adjoint (with respect to the Hilbert space structure of H^2) S^* is given by

$$S^* f = \frac{f - f(0)}{z}, \qquad f \in H^2.$$

A crucial fact is that $\mathcal{H}(b)$ remains invariant under the backward shift operator S^*, and we have the bounded linear operator $X_b : \mathcal{H}(b) \to \mathcal{H}(b)$ defined by

$$X_b = S^*|_{\mathcal{H}(b)}.$$

A central role in the theory of $\mathcal{H}(b)$ spaces is played by this operator. In particular, it is known that X_b is a contraction on $\mathcal{H}(b)$ [9, Theorem 18.7]. It should also be noted that a de Branges-Rovnyak space $\mathcal{H}(b)$ is not uniquely determined by the symbol b, that is, two different b's could define the same

$\mathcal{H}(b)$ (with a different norm). There exists a characterization of when two de Branges-Rovnyak spaces coincide (with equivalent norms). See [8, 9, 14] for a comprehensive account of these spaces and the reference therein.

Next, we shift our attention to the other key object of interest in this paper. In [4], Cowen and Douglas initiated a systematic study of a class of bounded linear operators on a complex separable Hilbert space $\mathcal{H}$ that possess an open set Ω in $\mathbb{C}$ of eigenvalues with constant and finite multiplicity. More specifically, for a bounded linear operator $T : \mathcal{H} \longrightarrow \mathcal{H}$ (in short, $T \in \mathcal{B}(\mathcal{H})$) and a natural number n, we say that $T \in B_n(\Omega)$ if the following conditions are satisfied:

(i) $\dim(\ker(T - \omega I)) = n$ for all $\omega \in \Omega$.
(ii) $\overline{\operatorname{span}}\{\ker(T - \omega I) : \omega \in \Omega\} = \mathcal{H}$.
(iii) $\operatorname{ran}(T - \omega I) = \mathcal{H}$ for all $\omega \in \Omega$.

We also say that T is a *Cowen-Douglas class operator of rank* n on Ω. Cowen and Douglas proved that for a given $T \in B_n(\Omega)$, there exists a hermitian holomorphic vector bundle E_T of rank n over Ω (see Section 3). Of course, our main interest is in the case $T = X_b$, $b \in H_1^\infty$. More specifically, we seek to determine whether such an operator X_b belongs to the Cowen-Douglas class. The following theorem, a central result of this paper, provides a precise answer to this question. As it is often the case in the theory of de Branges-Rovnyak spaces, the answer depends whether b is an extreme point of H_1^∞ or not. Recall that an extreme point of H_1^∞ is a function in H_1^∞ that does not lie in any open line segment connecting two distinct points in H_1^∞ [8, page 214]. It is well-known that a function $b \in H_1^\infty$ is an extreme point if and only if [8, Theorem 6.7]

$$\int_{\mathbb{T}} \log(1 - |b(z)|^2)\, dm = -\infty.$$

We recall that a function $b \in H^\infty$ admits boundary values (in terms of radial limits) almost everywhere with respect to the standard Lebesgue measure m on $\mathbb{T}(= \partial\mathbb{D})$. This justifies the integrand function in the above integration. We can now state our first main result.

Theorem 1.1. *Let $b \in H_1^\infty$. Then the following statements hold:*

(i) *If b is an extreme point, then $X_b \notin B_n(\Omega)$ for any natural number n and open set Ω in $\mathbb{C}$.*
(ii) *If b is a non-extreme point, then $X_b \in B_1(\mathbb{D})$.*

This result has several significant implications. Here, we highlight two key points. First, it provides a wealth of new and exotic examples of Cowen-Douglas class operators of rank one. We now know that the backward shift operators on de Branges-Rovnyak spaces corresponding to non-extreme points indeed belong to the Cowen-Douglas class $B_1(\mathbb{D})$. This, in turn, creates a natural link between these two concepts.

Second, Theorem 1.1 will give us a method to determine whether two different operators X_{b_1} and X_{b_2}, acting respectively on two different de Branges-Rovnyak spaces $\mathcal{H}(b_1)$ and $\mathcal{H}(b_2)$, are unitary equivalent. Recall that $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$ are *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $UT_1 = T_2U$. We write this simply as

$$T_1 \cong T_2.$$

Given $T \in \mathcal{B}(\mathcal{H})$ which belongs to $B_1(\mathbb{D})$, we continue our discussion of the hermitian holomorphic vector bundle E_T over $\mathbb{D}$ (as Theorem 1.1 allows us to restrict our attention to the rank one case). In this case, the line bundle E_T yields the *curvature* $\mathcal{K}_T$ of T, where (see [4, Theorem 1.17])

$$\mathcal{K}_T(\omega) = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log \|\gamma_{T,\omega}\|^2 \qquad (\omega \in \mathbb{D}), \tag{1.1}$$

and $\gamma_{T,\cdot} : \mathbb{D} \longrightarrow \mathcal{H}$ is any nonzero holomorphic cross-section of E_T. The important fact is that this curvature is a complete unitary invariant for operators in $B_1(\mathbb{D})$. See Section 3 for details.

Given a non-extreme point $b \in H_1^\infty$, we consider the *Pythagorean pair* (a, b), where $a \in H_1^\infty$ is the unique outer function such that $a(0) > 0$ and

$$|a|^2 + |b|^2 = 1,$$

a.e. on $\mathbb{T}$. We set the ratio

$$\Phi_b := \frac{b}{a}.$$

Let $b_1, b_2 \in H_1^\infty$ be two non-extreme points. Assume that (a_1, b_1) and (a_2, b_2) are the corresponding Pythagorean pairs. In Theorem 3.3, we prove that

$$X_{b_1} \cong X_{b_2},$$

if and only if

$$\frac{|\Phi'_{b_1}(\omega)|}{1 + |\Phi_{b_1}(\omega)|^2} = \frac{|\Phi'_{b_2}(\omega)|}{1 + |\Phi_{b_2}(\omega)|^2} \qquad (\omega \in \mathbb{D}).$$

In the context of non-extreme case, there is another operator that interests us. If $b \in H_1^\infty$ is a non-extreme point, then $S\mathcal{H}(b) \subseteq \mathcal{H}(b)$, and consequently, the restriction operator

$$S_b = S|_{\mathcal{H}(b)}$$

is a bounded operator on $\mathcal{H}(b)$, that is $S_b \in \mathcal{B}(\mathcal{H}(b))$. In Theorem 2.2, we prove that

$$S_b^* \in B_1(\mathbb{D}).$$

Using one more time the fact that the curvature defined in (1.1) is a complete unitary invariant, we get, in Theorem 3.4, that for non extreme points $b_1, b_2 \in H_1^\infty$,

$$S_{b_1} \cong S_{b_2}$$

if and only if

$$\frac{|b_1'(\omega)|}{1-|b_1(\omega)|^2} = \frac{|b_2'(\omega)|}{1-|b_2(\omega)|^2} \qquad (\omega \in \mathbb{D}).$$

We establish an additional connection between classical Carathéodory angular derivatives, de Branges Rovnyak spaces, and the Cowen-Douglas class of operators as follows (see Theorem 4.1): Let $b_1, b_2 \in H_1^\infty$ be two rational (not inner) functions in H_1^∞. Assume that

$$S_{b_1} \cong S_{b_2}.$$

Then b_1 and b_2 must have the same Carathéodory angular derivative points.

The rest of the paper is organized as follows. Section 2 provides an overview of de Branges-Rovnyak spaces and highlights that operators corresponding to non-extreme points belong only to $B_1(\mathbb{D})$. Section 3 computes the curvatures of operators associated with de Branges-Rovnyak spaces and presents a clear-cut expression for complete unitary invariants. Section 4 connects our theory with yet another classical notion, namely Carathéodory angular derivatives. The final section, Section 5, provides a concrete example to illustrate some of the main results of this paper.

2. de Branges-Rovnyak spaces in $B_1(\mathbb{D})$

Generally, the theory of de Branges-Rovnyak spaces bifurcates into two directions depending on whether the symbol $b \in H_1^\infty$ is an extreme point or not. The purpose of this section is to prove the central result (Theorem 1.1) of this paper, which once more explicitly establishes this dichotomy.

We begin by recalling some fundamental and well-known concepts concerning de Branges-Rovnyak spaces. First, the Hardy space H^2 is the Hilbert space of holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_2 := \sup_{r\in(0,1)} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta\right)^{\frac{1}{2}} < \infty. \tag{2.1}$$

As noted in the introduction, the de Branges-Rovnyak space $\mathcal{H}(b)$ associated with $b \in H_1^\infty$ is given by the range space

$$\mathcal{H}(b) = (I - T_b T_{\bar{b}})^{\frac{1}{2}} H^2.$$

This space is an algebraic complement of the range space bH^2 (a Hilbert space with respect to a similar range-based inner product). It is well known that $\mathcal{H}(b)$ is a reproducing kernel Hilbert space corresponding to the kernel function $k^b : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ defined by

$$k^b(z, \omega) = k^b_\omega(z) = \frac{1 - \overline{b(\omega)}b(z)}{1 - \overline{\omega}z},$$

for all $z, \omega \in \mathbb{D}$. In particular, for every function $f \in \mathcal{H}(b)$ and $\omega \in \mathbb{D}$, we have the *reproducing property* as

$$f(\omega) = \langle f, k^b_\omega \rangle_b,$$

and $\{k^b_\omega : \omega \in \mathbb{D}\}$ is a total set in $\mathcal{H}(b)$. It is worth pointing out that the kernel function k^b_ω is a scaling of the Szegö kernel k_ω of the disc $\mathbb{D}$, where $k_\omega(z) = (1 - \overline{\omega}z)^{-1}$ for all $z, \omega \in \mathbb{D}$. More specifically, given that $T_{\bar{b}}k_\omega = \overline{b(\omega)}k_\omega$, we have

$$k^b_\omega = (I - T_b T_{\bar{b}})k_\omega,$$

for all $\omega \in \mathbb{D}$. Also, recall that

$$X_b f = S^* f \qquad (f \in \mathcal{H}(b)),$$

defines a contraction on $\mathcal{H}(b)$ [9, Theorem 18.7], where S^* is the backward shift operator on H^2 given by

$$S^* f = T_{\bar{z}} f = \frac{f - f(0)}{z}.$$

In view of the operator X_b, we can define a more general operator Q_w for each $w \in \mathbb{D}$ as follows:

$$Q_w f(z) = \frac{f(z) - f(\omega)}{z - \omega},$$

for all $f \in \mathcal{H}(b)$ and $z \in \mathbb{D}$. Observe that, since X_b is a contraction, the operator $I - \omega X_b$ is invertible for every $\omega \in \mathbb{D}$, and

$$Q_w f = (I - \omega X_b)^{-1} X_b,$$

whence $Q_\omega f \in \mathcal{H}(b)$ for all $f \in \mathcal{H}(b)$. Moreover, note that $Q_0 = X_b$.

Returning to X_b, we observe that $S^* b \in \mathcal{H}(b)$, and for every $f \in \mathcal{H}(b)$, we have the well-known identity [9, Theorem 18.22]

$$X_b^* f = Sf - \langle f, S^* f \rangle_b b. \tag{2.2}$$

When $b \in H^\infty_1$ is a non-extreme point, it follows that $b \in \mathcal{H}(b)$, and then, the above identity implies that [9, Corollary 23.9 and Theorem 24.1]

$$S\mathcal{H}(b) \subseteq \mathcal{H}(b).$$

In this case, we write the restriction operator

$$S_b = S|_{\mathcal{H}(b)}.$$

Then (2.2) can be written as

$$X_b^* = S_b - b \otimes S^* b. \tag{2.3}$$

In view of the characterizations of extreme points introduced in Section 1, we also recall that a function $b \in H_1^\infty$ is non-extreme if and only if

$$\int_{\mathbb{T}} \log(1 - |b(z)|^2)\, dm > -\infty.$$

The following lemma will be the key point to studying the membership of X_b to the Cowen-Douglas class.

Lemma 2.1. *Let b be a non-extreme point of H_1^∞ and let $\omega \in \mathbb{D}$. Then $S_b - \omega I$ is a Fredholm operator with*

$$\operatorname{Ind}(S_b - \omega I) = -1.$$

Proof. By the definition of S_b, it is clear that $S_b f = wf$ for $f \in \mathcal{H}(b)$ implies that $f = 0$. This yields $\ker(S_b - \omega I) = \{0\}$ (that is, $\sigma_p(S_b) = \emptyset$). Let us prove now that

$$\operatorname{ran}(S_b - \omega I) = (\mathbb{C}k_\omega^b)^\perp.$$

First, let $f \in \operatorname{ran}(S_b - \omega I)$, that is,

$$f = (z - \omega)g,$$

for some $g \in \mathcal{H}(b)$. Then the reproducing property implies

$$\langle f, k_\omega^b \rangle_b = f(\omega) = 0,$$

which proves that $f \in (\mathbb{C}k_\omega^b)^\perp$. Conversely, let $f \perp k_\omega^b$ for some $f \in \mathcal{H}(b)$. Using the reproducing property, we have $f(\omega) = 0$. We know, in general, that $Q_\omega f \in \mathcal{H}(b)$, where, in this particular situation, we have

$$(Q_\omega f)(z) = \frac{f(z) - f(\omega)}{z - \omega} = \frac{f(z)}{z - \omega},$$

for all $z \in \mathbb{D}$. This implies

$$f = (S_b - \omega I) Q_\omega f.$$

Thus we deduce that $(\mathbb{C}k_\omega^b)^\perp \subseteq \operatorname{ran}(S_b - \omega I)$, proving the reversed inclusion. In particular, $\operatorname{ran}(S_b - \omega I) = (\mathbb{C}k_\omega^b)^\perp$ proves that $\operatorname{ran}(S_b - \omega I)$ is closed and

$$\ker(S_b^* - \overline{\omega} I) = (\operatorname{ran}(S_b - \omega I))^\perp = \mathbb{C}k_\omega^b. \tag{2.4}$$

We thus conclude that $S_b - \omega I$ is Fredholm, with index equals to

$$\operatorname{Ind}(S_b - \omega I) = \dim(\ker(S_b - \omega I)) - \dim(\ker(S_b^* - \overline{\omega} I)) = -1.$$

This concludes the proof of the lemma. □

This result may be of independent interest. With this in place, we are now ready to prove the central result of this paper:

Proof of Theorem 1.1: To prove (i), we assume that $b \in H_1^\infty$ is an extreme point. It follows from [9, Theorem 26.1] that the point spectrum of X_b is given by

$$\sigma_p(X_b) = \{\overline{\omega} : \omega \in \mathbb{D},\, b(\omega) = 0\}.$$

In particular, if $X_b \in B_n(\Omega)$ for some $n \geqslant 1$ and some nonempty open subset Ω of $\mathbb{D}$, then

$$b|_\Omega \equiv 0.$$

By continuation principle, we thus deduce that $b \equiv 0$, which contradicts the fact that the function

$$\log(1 - |b|^2) \notin L^1(\mathbb{T}).$$

For (ii), assume that $b \in H_1^\infty$ is not an extreme point. In this case, we recall from [9, Theorem 24.13] that

$$\sigma_p(X_b) = \mathbb{D},$$

and

$$\ker(X_b - \overline{\omega} I) = \mathbb{C} k_\omega, \tag{2.5}$$

for all $\omega \in \mathbb{D}$. In particular, $\dim(\ker(X_b - \overline{\omega} I)) = 1$ for every $\omega \in \mathbb{D}$. Moreover, we recall from [9, Corollary 23.26] that

$$\overline{\text{span}}\{\ker(X_b - \overline{\omega} I) : \omega \in \mathbb{D}\} = \overline{\text{span}}\{k_\omega : \omega \in \mathbb{D}\} = \mathcal{H}(b).$$

Thus, to prove that $X_b \in B_1(\mathbb{D})$, it only remains to verify that

$$\text{ran}(X_b - \overline{\omega} I) = \mathcal{H}(b),$$

for all $\omega \in \mathbb{D}$. Equivalently,

$$\text{ran}(X_b^* - \omega I) \text{ is closed},$$

and

$$\ker(X_b^* - \omega I) = \{0\},$$

for all $\omega \in \mathbb{D}$. Fix $\omega \in \mathbb{D}$. Note that (2.3) implies that

$$X_b^* - \omega I = S_b - \omega I - b \otimes S^* b.$$

According to Lemma 2.1, $S_b - \omega I$ is a Fredholm operator of index -1. Thus $X_b^* - \omega I$ is also a Fredholm operator with index -1. But, by (2.5), we have

$$\dim(\ker(X_b - \overline{\omega} I)) = 1.$$

Hence $\ker(X_b^* - \omega I) = \{0\}$ and $\text{ran}(X_b^* - \omega I)$ is closed, which proves that

$$\text{ran}(X_b - \overline{\omega} I) = \mathcal{H}(b).$$

Finally, we can conclude that $X_b \in B_1(\mathbb{D})$. □

In particular, this result exhibits yet another property of extreme and non-extreme points of H_1^∞. We recall once again that for a non-extreme $b \in H_1^\infty$, we have $S\mathcal{H}(b) \subseteq \mathcal{H}(b)$, and we define the restriction operator

$$S_b = S|_{\mathcal{H}(b)}.$$

We claim that S_b^* is in $B_1(\mathbb{D})$:

Theorem 2.2. *Let b be a non-extreme point of H_1^∞. Then $S_b^* \in B_1(\mathbb{D})$.*

Proof. Recall that (2.4) says that, for every $\omega \in \mathbb{D}$, we have

$$\ker(S_b^* - \overline{\omega} I) = \mathbb{C} k_\omega^b.$$

Hence

$$\dim(\ker(S_b^* - \overline{\omega} I)) = 1,$$

and we also trivially get that

$$\overline{\text{span}}\{\ker(S_b^* - \overline{\omega} I) : \omega \in \mathbb{D}\} = \mathcal{H}(b).$$

On the other hand, according to Lemma 2.1, $S_b - \omega I$ and thus $S_b^* - \overline{\omega} I$ has a closed range, which finally proves that $S_b^* \in B_1(\mathbb{D})$. □

A large and important class of extreme points of H_1^∞ are the inner functions. If $b = \Theta$ is an inner function, then $\mathcal{H}(b)$ coincides with the model space

$$K_\Theta = H^2 \ominus \Theta H^2,$$

a prototype of the family of closed backward shift invariant subspaces of H^2. In this case, X_b becomes the adjoint of the model operator S_Θ, where

$$S_\Theta = P_{K_\Theta} T_z|_{K_\Theta},$$

and P_{K_Θ} is the orthogonal projection of H^2 onto K_Θ (see [13] on model operators). Theorem 1.1, in particular, proves that model operators' adjoints are not in $B_n(\mathbb{D})$ for any $n \in \mathbb{N}$. In the context of Sz.-Nagy and Foias models and de Branges–Rovnyak spaces with non-extreme symbols, we refer the reader to [12].

3. Curvatures as unitary invariants

Operators in $B_n(\Omega)$ are naturally associated with curvature (in view of the Chern connection). In our case, we already know that X_b belongs to the Cowen-Douglas class if and only if b is a non-extreme point, and in that case, X_b is in the Cowen-Douglas class of rank one (that is, $X_b \in B_1(\mathbb{D})$). However, for the rank-one case, curvature serves as a complete unitary invariant (see Theorem 3.1 below). This motivates our study in this section, where we compute the curvature of the line bundle E_{X_b} corresponding to

$X_b \in B_1(\mathbb{D})$. Using Pythagorean pairs corresponding to non-extreme points, we provide computable unitary invariants for X_b.

To construct the curvature, we first isolate the hermitian holomorphic vector bundle corresponding to a given $T \in \mathcal{B}(\mathcal{H})$ which belongs to $B_1(\mathbb{D})$. Given $\omega \in \mathbb{D}$, we define

$$E_T(\omega) = \{\omega\} \times \ker(T - \omega I).$$

Then $\omega \mapsto E_T(\omega)$ defines a rank one hermitian holomorphic vector bundle over $\mathbb{D}$ as (see [4])

$$\begin{array}{c} E_T := \bigcup\limits_{\omega \in \mathbb{D}} E_T(\omega) \\ \Big\downarrow \\ \mathbb{D} \end{array}$$

In this case, there exists a holomorphic $\mathcal{H}$-valued function $\gamma_{T,\cdot} : \mathbb{D} \longrightarrow \mathcal{H}$ such that for every $\omega \in U$, we have the spanning property

$$\mathbb{C}\gamma_{T,\omega} = \ker(T - \omega I).$$

The *curvature* $\mathcal{K}_T$ of T is then given by

$$\mathcal{K}_T(\omega) = -\frac{\partial^2}{\partial \omega \partial \overline{\omega}} \log \|\gamma_{T,w}\|^2 \qquad (\omega \in \mathbb{D}).$$

The important fact is that this curvature is a complete unitary invariant for operators in $B_1(\mathbb{D})$ (see Cowen and Douglas [4]):

Theorem 3.1. *Let Ω be a domain in $\mathbb{C}$, $\mathcal{H}_j$ be a Hilbert space, and let $T_j \in \mathcal{B}(\mathcal{H}_j)$, $j = 1, 2$. Assume that $T_1, T_2 \in B_1(\Omega)$. Then $T_1 \cong T_2$ if and only if*

$$\mathcal{K}_{T_1} = \mathcal{K}_{T_2} \text{ on } \Omega.$$

Our primary interest lies in the case $T = X_b$. By Theorem 1.1, we know that when b is a non-extreme point in H_1^∞, then $X_b \in B_1(\mathbb{D})$. Recall from (2.5) that $\ker(X_b - \omega I) = \mathbb{C}k_{\overline{\omega}}$, and hence

$$\gamma_{X_b,\omega} = k_{\overline{\omega}} \in \mathcal{H}(b), \tag{3.1}$$

for all $\omega \in \mathbb{D}$. To get a criterion for unitarily equivalence between two X_b operators, we will need the following simple technical lemma.

Lemma 3.2. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{C}$ be a holomorphic function. Then, for every $\omega \in \mathbb{D}$, we have*

$$\frac{\partial^2}{\partial \omega \partial \overline{\omega}} \log(1 + |\varphi(\omega)|^2) = \frac{|\varphi'(\omega)|^2}{(1 + |\varphi(\omega)|^2)^2}.$$

Moreover, if $\varphi(\mathbb{D}) \subseteq \mathbb{D}$*, we also have, for every* $\omega \in \mathbb{D}$*,*

$$\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log(1 - |\varphi(\omega)|^2) = -\frac{|\varphi'(\omega)|^2}{(1 - |\varphi(\omega)|^2)^2}.$$

Proof. Since φ is analytic, we have

$$\frac{\partial}{\partial\overline{\omega}} \log(1 + |\varphi(\omega)|^2) = \frac{\partial}{\partial\overline{\omega}} \log(1 + \varphi(\omega)\overline{\varphi(\omega)}) = \frac{\overline{\varphi'(\omega)}\varphi(\omega)}{1 + \overline{\varphi(\omega)}\varphi(\omega)}.$$

Thus

$$\begin{aligned}\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log(1 + |\varphi(\omega)|^2) =& \frac{\partial}{\partial\omega}\left(\frac{\overline{\varphi'(\omega)}\varphi(\omega)}{1 + \overline{\varphi(\omega)}\varphi(\omega)}\right) \\ =& \frac{\overline{\varphi'(\omega)}\varphi'(\omega)(1 + \overline{\varphi(\omega)}\varphi(\omega)) - \overline{\varphi'(\omega)}\varphi(\omega)\overline{\varphi(\omega)}\varphi'(\omega)}{(1 + \overline{\varphi(\omega)}\varphi(\omega))^2} \\ =& \frac{|\varphi'(\omega)|^2 + |\varphi'(\omega)|^2|\varphi(\omega)|^2 - |\varphi'(\omega)|^2|\varphi(\omega)|^2}{(1 + |\varphi(\omega)|^2)^2} \\ =& \frac{|\varphi'(\omega)|^2}{(1 + |\varphi(\omega)|^2)^2}.\end{aligned}$$

One can prove the second formula similarly. □

Recall that when $b \in H_1^\infty$ is not an extreme point, then there exists a unique outer function a in H^∞ such that

$$a(0) > 0,$$

and

$$|a|^2 + |b|^2 = 1 \text{ a.e. on } \mathbb{T}.$$

The pair (a, b) is called a *Pythagorean pair.* Given a Pythagorean pair (a, b), if b is not an extreme point of H_1^∞, then (see [9, Theorem 23.23])

$$k_\omega \in \mathcal{H}(b),$$

and (see [9, Corollary 23.25])

$$\|k_\omega\|_b^2 = \left(1 + \frac{|b(\omega)|^2}{|a(\omega)|^2}\right) \cdot \frac{1}{1 - |\omega|^2}, \tag{3.2}$$

for all $\omega \in \mathbb{D}$. We are now ready to present the complete unitary invariants for the model operators acting on de Branges-Rovnyak spaces through the lens of the Cowen-Douglas class.

Theorem 3.3. *Let* $b_1, b_2 \in H_1^\infty$ *be two non-extreme points. Assume that* (a_1, b_1) *and* (a_2, b_2) *are two Pythagorean pairs, and let*

$$\Phi_j = \frac{b_j}{a_j},$$

for $j = 1, 2$. Then

$$X_{b_1} \cong X_{b_2},$$

if and only if for every $\omega \in \mathbb{D}$, we have

$$\frac{|\Phi_1'(\omega)|}{1 + |\Phi_1(\omega)|^2} = \frac{|\Phi_2'(\omega)|}{1 + |\Phi_2(\omega)|^2},$$

Proof. As already pointed out, Theorem 1.1 implies that $X_{b_1}, X_{b_2} \in B_1(\mathbb{D})$. Moreover, by (2.5), for every $\omega \in \mathbb{D}$, we have

$$\ker(X_{b_1} - \omega I) = \ker(X_{b_2} - \omega I) = \mathbb{C} k_{\overline{\omega}}.$$

Now it follows from Theorem 3.1 that $X_{b_1} \cong X_{b_2}$ if and only if for every $\omega \in \mathbb{D}$, we have

$$\mathcal{K}_{X_{b_1}}(\omega) = \mathcal{K}_{X_{b_2}}(\omega),$$

where, according to (3.1), we have

$$\mathcal{K}_{X_{b_i}}(\omega) = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log \|k_{\overline{\omega}}\|_{b_i}^2 = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log \|k_{\omega}\|_{b_i}^2,$$

for $i = 1, 2$. But according to (3.2), we have

$$\|k_\omega\|_{b_i}^2 = (1 + |\Phi_i(\omega)|^2) \cdot \frac{1}{1 + |\omega|^2},$$

whence

$$\log \|k_\omega\|_{b_i}^2 = \log(1 + |\Phi(\omega)|^2) - \log(1 - |\omega|^2),$$

for all $\omega \in \mathbb{D}$. It follows from Lemma 3.2, for $i = 1, 2$, and for all $\omega \in \mathbb{D}$, that

$$\mathcal{K}_{X_{b_i}}(\omega) = -\frac{|\Phi_i'(\omega)|^2}{(1 + |\Phi_i(\omega)|^2)^2} + \frac{1}{(1 - |\omega|^2)^2}.$$

Therefore $X_{b_1} \cong X_{b_2}$ if and only if

$$-\frac{|\Phi_1'(\omega)|^2}{(1 + |\Phi_1(\omega)|^2)^2} + \frac{1}{(1 - |\omega|^2)^2} = -\frac{|\Phi_2'(\omega)|^2}{(1 + |\Phi_2(\omega)|^2)^2} + \frac{1}{(1 - |\omega|^2)^2},$$

or equivalently,

$$\frac{|\Phi_1'(\omega)|^2}{(1 + |\Phi_1(\omega)|^2)^2} = \frac{|\Phi_2'(\omega)|^2}{(1 + |\Phi_2(\omega)|^2)^2},$$

for all $\omega \in \mathbb{D}$. This completes the proof of the theorem. □

Next, we turn to S_b for a non-extreme point $b \in H_1^\infty$. By Theorem 2.2, we know that $S_b^* \in B_1(\mathbb{D})$. This raises the question of the unitary equivalence of a pair of such operators. The following is our answer:

Theorem 3.4. *Let $b_1, b_2 \in H_1^\infty$ be two non-extreme points. Then*

$$S_{b_1} \cong S_{b_2},$$

if and only if for every $\omega \in \mathbb{D}$, we have

$$\frac{|b_1'(\omega)|}{1-|b_1(\omega)|^2} = \frac{|b_2'(\omega)|}{1-|b_2(\omega)|^2}. \tag{3.3}$$

Proof. We prove the equivalent version that $S_{b_1}^* \cong S_{b_2}^*$. According to Theorem 2.2, $S_{b_1}^*$ and $S_{b_2}^*$ are in $B_1(\mathbb{D})$, and we can apply Theorem 3.1 which implies that $S_{b_1}^* \cong S_{b_2}^*$ is equivalent to

$$\mathcal{K}_{S_{b_1}^*} = \mathcal{K}_{S_{b_2}^*} \text{ on } \mathbb{D}.$$

It follows from (2.4) that, for $i = 1, 2$, we have

$$\ker(S_{b_i}^* - \omega I) = \mathbb{C}k_{\overline{\omega}}^{b_i},$$

whence

$$\mathcal{K}_{S_{b_i}^*}(\omega) = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log \|k_{\overline{\omega}}^{b_i}\|_{b_i}^2 = -\frac{\partial^2}{\partial\omega\partial\overline{\omega}} \log \|k_{\omega}^{b_i}\|_{b_i}^2,$$

for all $\omega \in \mathbb{D}$. But

$$\|k_{\omega}^{b_i}\|_{b_i}^2 = \frac{1-|b_i(\omega)|^2}{1-|\omega|^2},$$

and then

$$\log \|k_{\omega}^{b_i}\|_{b_i}^2 = \log(1-|b_i(\omega)|^2) - \log(1-|\omega|^2),$$

for all $\omega \in \mathbb{D}$. According to Lemma 3.2, we thus get

$$\mathcal{K}_{S_{b_i}^*}(\omega) = -\frac{|b_i'(\omega)|^2}{(1-|b_i(\omega)|^2)^2} + \frac{1}{(1+|\omega|^2)^2},$$

for all $\omega \in \mathbb{D}$. Therefore $S_{b_1}^* \cong S_{b_2}^*$ is equivalent to

$$-\frac{|b_1'(\omega)|^2}{(1-|b_1(\omega)|^2)^2} + \frac{1}{(1+|\omega|^2)^2} = -\frac{|b_2'(\omega)|^2}{(1-|b_2(\omega)|^2)^2} + \frac{1}{(1+|\omega|^2)^2},$$

that is,

$$\frac{|b_1'(\omega)|}{1-|b_1(\omega)|^2} = \frac{|b_2'(\omega)|}{1-|b_2(\omega)|^2},$$

for all $\omega \in \mathbb{D}$. This completes the proof of the theorem. □

For the class of operators X_b, $b \in H_1^\infty$ non-extreme, the complete unitary invariants obtained in this section appear to be the first of their kind. A similar conclusion holds for the class of operators S_b.

4. Carathéodory angular derivatives

Recently, there has been particular interest in the case where b is a rational (not inner) function in H_1^∞. Indeed, in that case, we have a nice description of the space $\mathcal{H}(b)$. See for instance [2, 5, 6, 7]. In this particular case, we prove that condition (3.3) appearing in Theorem 3.4 is connected to Carathéodory angular derivatives. Recall that a function $b \in H_1^\infty$ has a Carathéodory angular derivative (an ADC for short) at a point $\zeta \in \mathbb{T}$ if b and b' have a non-tangential limit at ζ, with

$$|b(\zeta)| = 1.$$

A well known result of Carathéodory says that b has an ADC at ζ if and only if

$$c = \liminf_{z \to \zeta} \frac{1 - |b(z)|^2}{1 - |z|^2} < \infty.$$

Moreover, in that case, we have (cf. [9, Theorem 21.1])

$$c > 0,$$

and

$$c = \zeta b'(\zeta)\overline{b(\zeta)}.$$

The following theorem is essentially in the context of Theorem 3.4. More specifically, we apply the identity given in (3.3).

Theorem 4.1. *Let b_1 and b_2 be two rational (not inner) functions in H_1^∞. Assume that*

$$S_{b_1} \cong S_{b_2}.$$

Then b_1 and b_2 must have the same Carathéodory angular derivative points.

Proof. It is known (see [5, Lemma 3.1]) that b_1 and b_2 are non extreme points in H_1^∞. Then, according to Theorem 3.4, for every $\omega \in \mathbb{D}$, we have

$$\frac{|b_1'(\omega)|}{1 - |b_1(\omega)|^2} = \frac{|b_2'(\omega)|}{1 - |b_2(\omega)|^2}. \tag{4.1}$$

Assume now that $\zeta \in \mathbb{T}$ is a point where b_1 has a Carathéodory angular derivative. This means that there exists a sequence $(\omega_n)_n$ in $\mathbb{D}$ such that $\omega_n \to \zeta$ as $n \to \infty$, and

$$\lim_{n \to \infty} \frac{1 - |b_1(\omega_n)|^2}{1 - |\omega_n|^2} = c = |b_1'(\zeta)| > 0.$$

Using the fact that b_1' is continuous on the closed unit disk, this implies that

$$\frac{1 - |\omega_n|^2}{1 - |b_1(\omega_n)|^2} |b_1'(\omega_n)|^2 \longrightarrow \frac{1}{c} |b_1'(\zeta)| = 1, \qquad \text{as } n \to \infty.$$

Hence, it follows from (4.1) that

$$\frac{1-|\omega_n|^2}{1-|b_2(\omega_n)|^2}|b_2'(\omega_n)|^2 \longrightarrow 1, \qquad \text{as } n\to\infty.$$

But, since b_2 is a rational function in H_1^∞, the function b_2' is also bounded, and then for n sufficiently large, we have

$$\frac{1}{2} \leqslant \frac{1-|\omega_n|^2}{1-|b_2(\omega_n)|^2}|b_2'(\omega_n)| \leqslant \frac{1-|\omega_n|^2}{1-|b_2(\omega_n)|^2}\|b_2'\|_\infty.$$

In particular,

$$\liminf_{n\to+\infty} \frac{1-|b_2(\omega_n)|^2}{1-|\omega_n|^2} < \infty,$$

which gives that b_2 has a Carathéodory angular derivative at ζ. The above argument is of course symmetric proving the result. □

In this way, we have connected Carathéodory angular derivatives with Cowen-Douglas class operators and their curvatures.

5. An example

This section aims to illustrate the unitary equivalance results with even a more concrete example. Define $b_1, b_2 \in H_1^\infty$ by

$$b_1(z) = \frac{1+z}{2} \text{ and } b_2(z) = \frac{1-z}{2},$$

for $z\in\mathbb{D}$. It is easy to see that b_1 and b_2 are non extreme points. Moreover, (b_1, b_2) and (b_2, b_1) are Pythagorean pairs. Following the notations of Theorem 3.3, we have

$$\Phi_1(z) = \frac{b_1(z)}{b_2(z)} = \frac{1+z}{1-z},$$

and

$$\Phi_2(z) = \frac{b_2(z)}{b_1(z)} = \frac{1-z}{1+z}.$$

As $\Phi_1'(z) = \frac{2}{(1-z)^2}$, it follows that

$$\frac{|\Phi_1'(z)|}{1+|\Phi_1(z)|^2} = \frac{2}{|1-z|^2+|1+z|^2} = \frac{1}{1+|z|^2},$$

for all $z\in\mathbb{D}$. Similarly, $\Phi_2'(z) = -\frac{2}{(1+z)^2}$ implies

$$\frac{|\Phi_2'(z)|}{1+|\Phi_2(z)|^2} = \frac{2}{|1-z|^2+|1+z|^2} = \frac{1}{1+|z|^2},$$

for all $z\in\mathbb{D}$. Therefore, for every $z\in\mathbb{D}$,

$$\frac{|\Phi_1'(z)|}{1+|\Phi_1(z)|^2} = \frac{|\Phi_2'(z)|}{1+|\Phi_2(z)|^2},$$

and Theorem 3.3 implies that $X_{b_1} \cong X_{b_2}$.

In this example, we also note that

$$\mathcal{H}(b_1) \neq \mathcal{H}(b_2).$$

Indeed, since b_1 and b_2 are two non-extreme points of H_1^∞, it follows from [3] (see also [9, Corollary 27.17]) that if $\mathcal{H}(b_1) = \mathcal{H}(b_2)$, then it would imply that

$$a_1 a_2^{-1} \in H^\infty.$$

But $a_1 = b_2$ and $a_2 = b_1$, and thus

$$\frac{a_1(z)}{a_2(z)} = \frac{b_2(z)}{b_1(z)} = \frac{1-z}{1+z}.$$

Clearly,

$$\frac{1-z}{1+z} \notin H^\infty.$$

Therefore $\mathcal{H}(b_1) \neq \mathcal{H}(b_2)$. Note that in this example, Corollary 4.1 implies that S_{b_1} and S_{b_2} are not unitarily equivalent. Indeed, in that case, it is easy to see that b_1 has a Carathéodory angular derivative only at point 1, whereas b_2 has a Carathéodory angular derivative only at point -1.

It would be fascinating to explore how the results of this paper extend to several variables or more general domains. While many significant results are known in this direction, particularly in the context of Cowen-Douglas class operators [11] and related theories in several variables [10], further progress is needed. This is especially true given the sophistication required in the study of de Branges-Rovnyak spaces in several variables (however, see [1]).

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